

Constructing New Braided T -categories over Monoidal Hom-Hopf Algebras

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Abstract Let $Aut_{mHH}(H)$ denote the set of all automorphisms of a monoidal Hopf algebra H with bijective antipode in the sense of Caenepeel and Goyvaerts [2] and let G be a crossed product group $Aut_{mHH}(H) \times Aut_{mHH}(H)$. The main aim of this paper is to provide new examples of braided T -category in the sense of Turaev [15]. For this purpose, we first introduce a class of new categories ${}_H\mathcal{MHYD}^H(A, B)$ of (A, B) -Yetter-Drinfeld Hom-modules with $A, B \in Aut_{mHH}(H)$. Then we construct a category $\mathcal{MHYD}(H) = \{{}_H\mathcal{MHYD}^H(A, B)\}_{(A, B) \in G}$ and show that such category forms a new braided T -category, generalizing the main constructions by Panaite and Staic [11]. Finally we compute an explicit new example of such braided T -categories.

Key words: Monoidal Hom-Hopf algebra; Braided T -category; Monoidal (A, B) -Yetter-Drinfeld Hom-module.

Mathematics Subject Classification: 16W30.

0. INTRODUCTION

Braided T -categories introduced by Turaev [15] are of interest due to their applications in homotopy quantum field theories, which are generalizations of ordinary topological quantum field theories. As such, they are interesting to different research communities in mathematical physics (see [5, 6, 14, 16, 17]). Although Yetter-Drinfeld modules over Hopf algebras provide examples of such braided T -categories, these are rather trivial. The wish to obtain more interesting homotopy quantum field theories provides a strong motivation to find new examples of braided T -categories.

The aim of this article is to construct new examples of braided T -categories. This is achieved by generalizing an existing construction by Panaite and Staic [11] that twists Yetter-Drinfeld modules over a Hopf algebra H by Hopf algebra automorphisms. We will generalize this construction to twisted Yetter-Drinfeld modules over so-called monoidal

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Hom-Hopf algebras, which are Hopf algebras in the Hom-category of a monoidal category (see [2]). We find a suitable generalisation of the notion of a twisted Yetter-Drinfeld module for this setting and obtain a category of twisted Yetter-Drinfeld modules that is a braided T -category in the sense of Turaev [15].

The article is organized as follows.

We will present the background material in Section 1. This section contains the relevant definitions on monoidal Hom-Hopf algebras and braided T -categories necessary for the understanding of the construction. In Section 2, we define the notion of a Yetter-Drinfeld module over a monoidal Hom-Hopf algebra that is twisted by two monoidal Hom-Hopf algebra automorphisms as well as the notion of a monoidal Hom-entwining structure and show how such monoidal Hom-entwining structures are obtained from automorphisms of monoidal Hom-Hopf algebras.

Section 3 first introduces the tensor product of twisted Yetter-Drinfeld Hom-modules and then shows that the twisted Yetter-Drinfeld Hom-modules form a braided T -category in the sense of Turaev [13]. At the end of the section, we give an example of a monoidal Hom-Hopf algebra, which can be viewed as a generalization of Sweedler's Hopf algebra. And furthermore, we compute an example of a twisted Yetter-Drinfeld module over a monoidal Hom-Hopf algebra.

1. PRELIMINARIES

Throughout, let k be a fixed field. Everything is over k unless otherwise specified. We refer the readers to the books of Sweedler [13] for the relevant concepts on the general theory of Hopf algebras. Let (C, Δ) be a coalgebra. We use the "sigma" notation for Δ as follows:

$$\Delta(c) = \sum c_1 \otimes c_2, \quad \forall c \in C.$$

1.1. Braided T -categories.

A *monoidal category* $\mathcal{C} = (\mathcal{C}, \mathbb{I}, \otimes, a, l, r)$ is a category \mathcal{C} endowed with a functor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ (the *tensor product*), an object $\mathbb{I} \in \mathcal{C}$ (the *tensor unit*), and natural isomorphisms a (the *associativity constraint*), where $a_{U,V,W} : (U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W)$ for all $U, V, W \in \mathcal{C}$, and l (the *left unit constraint*) where $l_U : \mathbb{I} \otimes U \rightarrow U$, r (the *right unit constraint*) where $r_U : U \otimes \mathbb{I} \rightarrow U$ for all $U \in \mathcal{C}$, such that for all $U, V, W, X \in \mathcal{C}$, the *associativity pentagon* $a_{U,V,W \otimes X} \circ a_{U \otimes V, W, X} = (U \otimes a_{V,W,X}) \circ a_{U, V \otimes W, X} \circ (a_{U,V,W} \otimes X)$ and $(U \otimes l_V) \circ (r_U \otimes V) = a_{U, \mathbb{I}, V}$ are satisfied. A monoidal category \mathcal{C} is *strict* when all the constraints are identities.

Let G be a group and let $\text{Aut}(\mathcal{C})$ be the group of invertible strict tensor functors from

\mathcal{C} to itself. A category \mathcal{C} over G is called a *crossed category* if it satisfies the following:

- ◆ \mathcal{C} is a monoidal category;
- ◆ \mathcal{C} is disjoint union of a family of subcategories $\{\mathcal{C}_\alpha\}_{\alpha \in G}$, and for any $U \in \mathcal{C}_\alpha$, $V \in \mathcal{C}_\beta$, $U \otimes V \in \mathcal{C}_{\alpha\beta}$. The subcategory \mathcal{C}_α is called the α th component of \mathcal{C} ;
- ◆ Consider a group homomorphism $\varphi : G \rightarrow \text{Aut}(\mathcal{C})$, $\beta \mapsto \varphi_\beta$, and assume that $\varphi_\beta(\varphi_\alpha) = \varphi_{\beta\alpha\beta^{-1}}$, for all $\alpha, \beta \in G$. The functors φ_β are called conjugation isomorphisms.

Furthermore, \mathcal{C} is called strict when it is strict as a monoidal category.

Left index notation: Given $\alpha \in G$ and an object $V \in \mathcal{C}_\alpha$, the functor φ_α will be denoted by ${}^V(\cdot)$, as in Turaev [15] or Zunino [18], or even ${}^\alpha(\cdot)$. We use the notation $\overline{V}(\cdot)$ for ${}^{\alpha^{-1}}(\cdot)$. Then we have ${}^V id_U = id_{VU}$ and ${}^V(g \circ f) = {}^V g \circ {}^V f$. Since the conjugation $\varphi : G \rightarrow \text{Aut}(\mathcal{C})$ is a group homomorphism, for all $V, W \in \mathcal{C}$, we have ${}^{V \otimes W}(\cdot) = {}^V({}^W(\cdot))$ and $\mathbb{I}(\cdot) = {}^V(\overline{V}(\cdot)) = \overline{V}({}^V(\cdot)) = id_{\mathcal{C}}$. Since, for all $V \in \mathcal{C}$, the functor ${}^V(\cdot)$ is strict, we have ${}^V(f \otimes g) = {}^V f \otimes {}^V g$, for any morphisms f and g in \mathcal{C} , and ${}^V \mathbb{I} = \mathbb{I}$.

A *braiding* of a crossed category \mathcal{C} is a family of isomorphisms $(c = c_{U,V})_{U,V \in \mathcal{C}}$, where $c_{U,V} : U \otimes V \rightarrow {}^U V \otimes U$ satisfying the following conditions:

a) For any arrow $f \in \mathcal{C}_\alpha(U, U')$ and $g \in \mathcal{C}(V, V')$,

$$(({}^\alpha g) \otimes f) \circ c_{U,V} = c_{U'V'} \circ (f \otimes g).$$

b) For all $U, V, W \in \mathcal{C}$, we have

$$c_{U \otimes V, W} = a_{U \otimes V, W, U, V} \circ (c_{U, V \otimes W} \otimes id_V) \circ a_{U, V, W, V}^{-1} \circ (\iota_U \otimes c_{V, W}) \circ a_{U, V, W},$$

$$c_{U, V \otimes W} = a_{U, V, U, W}^{-1} \circ (\iota_{(U \otimes V)} \otimes c_{U, W}) \circ a_{U, V, U, W} \circ (c_{U, V} \otimes \iota_W) \circ a_{U, V, W}^{-1},$$

where a is the natural isomorphisms in the tensor category \mathcal{C} .

c) For all $U, V \in \mathcal{C}$ and $\beta \in G$,

$$\varphi_\beta(c_{U,V}) = c_{\varphi_\beta(U), \varphi_\beta(V)}.$$

A crossed category endowed with a braiding is called a *braided T -category*.

1.2. Monoidal Hom-Hopf algebras.

Let $\mathcal{M}_k = (\mathcal{M}_k, \otimes, k, a, l, r)$ denote the usual monoidal category of k -vector spaces and linear maps between them. Recall from [2] that there is the *monoidal Hom-category* $\tilde{\mathcal{H}}(\mathcal{M}_k) = (\mathcal{H}(\mathcal{M}_k), \otimes, (k, id), \tilde{a}, \tilde{l}, \tilde{r})$, a new monoidal category, associated with \mathcal{M}_k as follows:

- The objects of $\mathcal{H}(\mathcal{M}_k)$ are couples (M, μ) , where $M \in \mathcal{M}_k$ and $\mu \in \text{Aut}_k(M)$, the set of all k -linear automomorphisms of M ;
- The morphism $f : (M, \mu) \rightarrow (N, \nu)$ in $\mathcal{H}(\mathcal{M}_k)$ is the k -linear map $f : M \rightarrow N$ in \mathcal{M}_k satisfying $\nu \circ f = f \circ \mu$, for any two objects $(M, \mu), (N, \nu) \in \mathcal{H}(\mathcal{M}_k)$;

- The tensor product is given by

$$(M, \mu) \otimes (N, \nu) = (M \otimes N, \mu \otimes \nu)$$

for any $(M, \mu), (N, \nu) \in \mathcal{H}(\mathcal{M}_k)$.

- The tensor unit is given by (k, id) ;
- The associativity constraint \tilde{a} is given by the formula

$$\tilde{a}_{M,N,L} = a_{M,N,L} \circ ((\mu \otimes id) \otimes \varsigma^{-1}) = (\mu \otimes (id \otimes \varsigma^{-1})) \circ a_{M,N,L},$$

for any objects $(M, \mu), (N, \nu), (L, \varsigma) \in \mathcal{H}(\mathcal{M}_k)$;

- The left and right unit constraint \tilde{l} and \tilde{r} are given by

$$\tilde{l}_M = \mu \circ l_M = l_M \circ (id \otimes \mu), \quad \tilde{r}_M = \mu \circ r_M = r_M \circ (\mu \otimes id)$$

for all $(M, \mu) \in \mathcal{H}(\mathcal{M}_k)$.

We now recall from [2] the following notions used later.

A *unital monoidal Hom-associative algebra* (a monoidal Hom-algebra in Proposition 2.1 of [2]) is a vector space A together with an element $1_A \in A$ and linear maps

$$m : A \otimes A \rightarrow A; \quad a \otimes b \mapsto ab, \quad \alpha \in Aut_k(A)$$

such that

$$\alpha(a)(bc) = (ab)\alpha(c), \tag{1. 1}$$

$$\alpha(ab) = \alpha(a)\alpha(b),$$

$$a1_A = 1_A a = \alpha(a), \tag{1. 2}$$

$$\alpha(1_A) = 1_A, \tag{1. 3}$$

for all $a, b, c \in A$.

Remark 1.1. (1) In the language of Hopf algebras, m is called the Hom-multiplication, α is the twisting automorphism and 1_A is the unit. Note that Eq.(1.1) can be rewritten as $a(b\alpha^{-1}(c)) = (\alpha^{-1}(a)b)c$. The monoidal Hom-algebra A with α will be denoted by (A, α) .

(2) Let (A, α) and (A', α') be two monoidal Hom-algebras. A monoidal Hom-algebra map $f : (A, \alpha) \rightarrow (A', \alpha')$ is a linear map such that $f \circ \alpha = \alpha' \circ f$, $f(ab) = f(a)f(b)$ and $f(1_A) = 1_{A'}$.

(3) The definition of monoidal Hom-algebras is different from the unital Hom-associative algebras in [9] and [10] in the following sense. The same twisted associativity condition (1.1) holds in both cases. However, the unitality condition in their notion is the usual untwisted one: $a1_A = 1_A a = a$, for any $a \in A$, and the twisting map α does not need to be monoidal (that is, (1.2) and (1.3) are not required).

A *counital monoidal Hom-coassociative coalgebra* is an object (C, γ) in the category $\tilde{\mathcal{H}}(\mathcal{M}_k)$ together with linear maps $\Delta : C \rightarrow C \otimes C$, $\Delta(c) = c_1 \otimes c_2$ and $\varepsilon : C \rightarrow k$ such that

$$\gamma^{-1}(c_1) \otimes \Delta(c_2) = \Delta(c_1) \otimes \gamma^{-1}(c_2), \tag{1. 4}$$

$$\Delta(\gamma(c)) = \gamma(c_1) \otimes \gamma(c_2), \quad (1.5)$$

$$\begin{aligned} c_1 \varepsilon(c_2) &= \gamma^{-1}(c) = \varepsilon(c_1) c_2, \\ \varepsilon(\gamma(c)) &= \varepsilon(c) \end{aligned} \quad (1.6)$$

for all $c \in C$.

Remark 1.2. (1) Note that (1.4) is equivalent to $c_1 \otimes c_{21} \otimes \gamma(c_{22}) = \gamma(c_{11}) \otimes c_{12} \otimes c_2$. Analogue to monoidal Hom-algebras, monoidal Hom-coalgebras will be short for counital monoidal Hom-coassociative coalgebras without any confusion.

(2) Let (C, γ) and (C', γ') be two monoidal Hom-coalgebras. A monoidal Hom-coalgebra map $f : (C, \gamma) \rightarrow (C', \gamma')$ is a linear map such that $f \circ \gamma = \gamma' \circ f$, $\Delta \circ f = (f \otimes f) \circ \Delta$ and $\varepsilon' \circ f = \varepsilon$.

A monoidal Hom-bialgebra $H = (H, \alpha, m, 1_H, \Delta, \varepsilon)$ is a bialgebra in the monoidal category $\tilde{\mathcal{H}}(\mathcal{M}_k)$. This means that $(H, \alpha, m, 1_H)$ is a monoidal Hom-algebra and $(H, \alpha, \Delta, \varepsilon)$ is a monoidal Hom-coalgebra such that Δ and ε are morphisms of algebras, that is, for all $h, g \in H$,

$$\Delta(hg) = \Delta(h)\Delta(g), \quad \Delta(1_H) = 1_H \otimes 1_H, \quad \varepsilon(hg) = \varepsilon(h)\varepsilon(g), \quad \varepsilon(1_H) = 1.$$

A monoidal Hom-bialgebra (H, α) is called a *monoidal Hom-Hopf algebra* if there exists a morphism (called antipode) $S : H \rightarrow H$ in $\tilde{\mathcal{H}}(\mathcal{M}_k)$ (i.e., $S \circ \alpha = \alpha \circ S$), which is the convolution inverse of the identity morphism id_H (i.e., $S * id = 1_H \circ \varepsilon = id * S$). Explicitly, for all $h \in H$,

$$S(h_1)h_2 = \varepsilon(h)1_H = h_1S(h_2).$$

Remark 1.3. (1) Note that a monoidal Hom-Hopf algebra is by definition a Hopf algebra in $\tilde{\mathcal{H}}(\mathcal{M}_k)$.

(2) Furthermore, the antipode of monoidal Hom-Hopf algebras has almost all the properties of antipode of Hopf algebras such as

$$S(hg) = S(g)S(h), \quad S(1_H) = 1_H, \quad \Delta(S(h)) = S(h_2) \otimes S(h_1), \quad \varepsilon \circ S = \varepsilon.$$

That is, S is a monoidal Hom-anti-(co)algebra homomorphism. Since α is bijective and commutes with S , we can also have that the inverse α^{-1} commutes with S , that is, $S \circ \alpha^{-1} = \alpha^{-1} \circ S$.

In the following, we recall the notions of actions on monoidal Hom-algebras and coactions on monoidal Hom-coalgebras.

Let (A, α) be a monoidal Hom-algebra. A *left (A, α) -Hom-module* consists of an object (M, μ) in $\tilde{\mathcal{H}}(\mathcal{M}_k)$ together with a morphism $\psi : A \otimes M \rightarrow M$, $\psi(a \otimes m) = a \cdot m$ such that

$$\alpha(a) \cdot (b \cdot m) = (ab) \cdot \mu(m), \quad \mu(a \cdot m) = \alpha(a) \cdot \mu(m), \quad 1_A \cdot m = \mu(m),$$

for all $a, b \in A$ and $m \in M$.

Monoidal Hom-algebra (A, α) can be considered as a Hom-module on itself by the Hom-multiplication. Let (M, μ) and (N, ν) be two left (A, α) -Hom-modules. A morphism $f : M \rightarrow N$ is called left (A, α) -linear if $f(a \cdot m) = a \cdot f(m)$, $f \circ \mu = \nu \circ f$. We denoted the category of left (A, α) -Hom modules by $\tilde{\mathcal{H}}(A\mathcal{M}_k)$.

Similarly, let (C, γ) be a monoidal Hom-coalgebra. A *right (C, γ) -Hom-comodule* is an object (M, μ) in $\tilde{\mathcal{H}}(\mathcal{M}_k)$ together with a k -linear map $\rho_M : M \rightarrow M \otimes C$, $\rho_M(m) = m_{(0)} \otimes m_{(1)}$ such that

$$\mu^{-1}(m_{(0)}) \otimes \Delta_C(m_{(1)}) = (m_{(0)(0)} \otimes m_{(0)(1)}) \otimes \gamma^{-1}(m_{(1)}), \quad (1.7)$$

$$\rho_M(\mu(m)) = \mu(m_{(0)}) \otimes \gamma(m_{(1)}), \quad m_{(0)}\varepsilon(m_{(1)}) = \mu^{-1}(m), \quad (1.8)$$

for all $m \in M$.

(C, γ) is a Hom-comodule on itself via the Hom-comultiplication. Let (M, μ) and (N, ν) be two right (C, γ) -Hom-comodules. A morphism $g : M \rightarrow N$ is called right (C, γ) -colinear if $g \circ \mu = \nu \circ g$ and $g(m_{(0)}) \otimes m_{(1)} = g(m)_{(0)} \otimes g(m)_{(1)}$. The category of right (C, γ) -Hom-comodules is denoted by $\tilde{\mathcal{H}}(\mathcal{M}^C)$.

Let (H, α) be a monoidal Hom-bialgebra. We now recall from [4] that a monoidal Hom-algebra (B, β) is called a *left H -Hom-module algebra*, if (B, β) is a left H -Hom-module with action \cdot obeying the following axioms:

$$h \cdot (ab) = (h_1 \cdot a)(h_2 \cdot b), \quad h \cdot 1_B = \varepsilon(h)1_B, \quad (1.9)$$

for all $a, b \in B, h \in H$.

Recall from [7] that a monoidal Hom-algebra (B, β) is called a *left H -Hom-comodule algebra*, if (B, β) is a left H -Hom-comodule with coaction ρ obeying the following axioms:

$$\rho(ab) = a_{(-1)}b_{(-1)} \otimes a_{(0)}b_{(0)}, \quad \rho(1_B) = 1_B \otimes 1_H,$$

for all $a, b \in B, h \in H$.

Let (H, m, Δ, α) be a monoidal Hom-bialgebra. Recall from ([4, 7]) that a *left-right Yetter-Drinfeld Hom-module* over (H, α) is the object (M, \cdot, ρ, μ) which is both in $\tilde{\mathcal{H}}(H\mathcal{M})$ and $\tilde{\mathcal{H}}(\mathcal{M}^H)$ obeying the compatibility condition:

$$h_1 \cdot m_{(0)} \otimes h_2 m_{(1)} = (\alpha(h_2) \cdot m)_{(0)} \otimes \alpha^{-1}(\alpha(h_2) \cdot m)_{(1)} h_1. \quad (1.10)$$

Remark 1.4. (1) The category of all left-right Yetter-Drinfeld Hom-modules is denoted by $\tilde{\mathcal{H}}(H\mathcal{YD}^H)$ with understanding morphism.

(2) If (H, α) is a monoidal Hom-Hopf algebra with a bijective antipode S , then the above equality is equivalent to

$$\rho(h \cdot m) = \alpha(h_{21}) \cdot m_{(0)} \otimes (h_{22}\alpha^{-1}(m_{(1)}))S^{-1}(h_1),$$

for all $h \in H$ and $m \in M$.

2. (A, B) -YETTER-DRINFELD HOM-MODULES

In this section, we define the notion of a Yetter-Drinfeld module over a monoidal Hom-Hopf algebra that is twisted by two monoidal Hom-Hopf algebra automorphisms as well as the notion of a monoidal Hom-entwining structure and show how such monoidal Hom-entwining structures are obtained from automorphisms of monoidal Hom-Hopf algebras.

In what follows, let (H, α) be a monoidal Hom-Hopf algebra with the bijective antipode S and let $\text{Aut}_{mHH}(H)$ denote the set of all automorphisms of a monoidal Hopf algebra H .

Definition 2.1. Let $A, B \in \text{Aut}_{mHH}(H)$. A left-right (A, B) -Yetter-Drinfeld Hom-module over (H, α) is a vector space M such that:

- (1) (M, \cdot, μ) is a left H -Hom-module;
- (2) (M, ρ, μ) is a right H -Hom-comodule;
- (3) ρ and \cdot satisfy the following compatibility condition:

$$\rho(h \cdot m) = \alpha(h_{21}) \cdot m_{(0)} \otimes (B(h_{22})\alpha^{-1}(m_{(1)}))A(S^{-1}(h_1)), \quad (2.1)$$

for all $h \in H$ and $m \in M$. We denote by ${}_H\mathcal{MHYD}^H(A, B)$ the category of left-right (A, B) -Yetter-Drinfeld Hom-modules, morphisms being H -linear H -colinear maps.

Remark 2.2. Note that, A and B are bijective, Hom-algebra morphisms, Hom-coalgebra morphisms, and commute with S and α .

Proposition 2.3. One has that Eq.(2.1) is equivalent to the following equation:

$$h_1 \cdot m_{(0)} \otimes B(h_2)m_{(1)} = \mu((h_2 \cdot \mu^{-1}(m))_{(0)}) \otimes (h_2 \cdot \mu^{-1}(m))_{(1)}A(h_1). \quad (2.2)$$

Proof. Eq.(2.1) \implies Eq.(2.2). We compute as follows

$$\begin{aligned} & \mu((h_2 \cdot \mu^{-1}(m))_{(0)}) \otimes (h_2 \cdot \mu^{-1}(m))_{(1)}A(h_1) \\ \stackrel{(2.1)}{=} & \mu(\alpha(h_{221}) \cdot \mu^{-1}(m)_{(0)}) \otimes ((B(h_{222})\alpha^{-2}(m_{(1)}))A(S^{-1}(h_{21})))A(h_1) \\ = & \mu(h_{12} \cdot \mu^{-1}(m)_{(0)}) \otimes ((B\alpha^{-2}(h_2)\alpha^{-2}(m_{(1)}))A(S^{-1}\alpha(h_{112})))A\alpha^2(h_{111}) \\ = & \mu(\alpha^{-1}(h_1)) \cdot \mu^{-1}(m)_{(0)} \otimes B(h_2)m_{(1)} = h_1 \cdot m_{(0)} \otimes B(h_2)m_{(1)}. \end{aligned}$$

For Eq.(2.2) \implies Eq.(2.1), we have

$$\begin{aligned} & \alpha(h_{21}) \cdot m_{(0)} \otimes (B(h_{22})\alpha^{-1}(m_{(1)}))A(S^{-1}(h_1)) \\ \stackrel{(2.2)}{=} & \mu((\alpha(h_{22}) \cdot \mu^{-1}(m))_{(0)}) \otimes \alpha^{-1}((\alpha(h_{22}) \cdot \mu^{-1}(m))_{(1)}A\alpha(h_{21}))AS^{-1}(h_1) \\ = & \mu((h_2 \cdot \mu^{-1}(m))_{(0)}) \otimes \alpha^{-1}((h_2 \cdot \mu^{-1}(m))_{(1)}A\alpha(h_{12}))AS^{-1}\alpha(h_{11}) \\ = & (\alpha(h_2) \cdot m)_{(0)} \otimes (h_2 \cdot \mu^{-1}(m))_{(1)}(A(h_{12})AS^{-1}(h_{11})) = (h \cdot m)_{(0)} \otimes (h \cdot m)_{(1)}. \end{aligned}$$

This finishes the proof. ■

Example 2.4. For $A = B = id_H$, we have ${}_H\mathcal{MHYD}^H(id, id) = \mathcal{H}({}_H\mathcal{YD}^H)$, the usual monoidal Yetter-Drinfeld Hom-module category. (see [4, 7]).

Example 2.5. (1) Take a non-trivial monoidal Hom-Hopf algebra isomorphism $B \in Aut_{mHH}(H)$. We define $(H_B, \alpha) = (H, \alpha)$ as k -vector spaces, and we can consider a right H -Hom-comodule structure on H_B via the Hom-comultiplication Δ and a left H -Hom-module structure on H_B as follows:

$$h \cdot y = (B(h_2)\alpha^{-1}(y))S^{-1}(\alpha(h_1)).$$

for all $h \in H, y \in H_B$. Then it is not hard to check that $H_B \in {}_H\mathcal{MHYD}^H(id, B)$.

More generally, if $A, B \in Aut_{mHH}(H)$, define $H_{(A,B)}$ as follows: $(H_{(A,B)}, \alpha) = (H, \alpha)$ as k -vector spaces, with right H -Hom-comodule structure via Hom-comultiplication and left H -Hom-module structure given by:

$$h \cdot x = (B(h_2)\alpha^{-1}(x))A(S^{-1}(\alpha(h_1))).$$

for all $h, x \in H$. It is straightforward to check that $H_{(A,B)} \in {}_H\mathcal{MHYD}^H(A, B)$.

(2) Recall from Example 3.5 in [3] that $(H_4 = k\{1, g, x, gx\}, \alpha, \Delta, \varepsilon, S)$ is a monoidal Hom-Hopf algebra, where the algebraic structure are given as follows:

- The multiplication " \circ " is given by

$$\begin{array}{llll} 1 \circ 1 = 1, & 1 \circ g = g, & 1 \circ x = cx, & 1 \circ gx = cgx, \\ g \circ 1 = g, & g \circ g = 1, & g \circ x = cgx, & g \circ gx = cx, \\ x \circ 1 = cx, & x \circ g = -cgx, & x \circ x = 0, & x \circ gx = 0, \\ gx \circ 1 = cgx, & gx \circ g = -cx, & gx \circ x = 0, & gx \circ gx = 0; \end{array}$$

- The automorphism α is given by

$$\alpha(1) = 1, \alpha(g) = g, \alpha(x) = cx, \alpha(gx) = cgx,$$

for all $0 \neq c \in k$;

- The comultiplication Δ is defined by

$$\begin{array}{ll} \Delta(1) = 1 \otimes 1, & \Delta(g) = g \otimes g, \\ \Delta(x) = c^{-1}(x \otimes 1) + c^{-1}(g \otimes x), & \Delta(gx) = c^{-1}(gx \otimes g) + c^{-1}(1 \otimes gx); \end{array}$$

- The counit ε is defined by

$$\varepsilon(1) = 1, \varepsilon(g) = 1, \varepsilon(x) = 0, \varepsilon(gx) = 0,$$

and

- The antipode S is given by

$$S(1) = 1, S(g) = g, S(x) = -gx, S(gx) = -x,$$

Still from Example 3.5 in [3] that we have the automorphism group of the monoidal Hom-Hopf algebra H_4 : $Aut_{mHH}(H_4) = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix} \mid 0 \neq \lambda \in k \right\}$

In what follows, we will give an explicit describe on the Yetter-Drinfeld Hom-modules given in Part (1) above for H_4 .

Let $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c' & 0 \\ 0 & 0 & 0 & c' \end{pmatrix}$. By Part (1), we can define $H_{4A} = (H_4, \alpha)$ as k -vector spaces, but with the right H_4 -Hom-comodule structure via Δ and the left H_4 -module structures as follows:

$$\begin{aligned} 1 \cdot 1 &= 1, & 1 \cdot g &= g, & 1 \cdot x &= cx, & 1 \cdot gx &= cgx, \\ g \cdot 1 &= 1, & g \cdot g &= g, & g \cdot x &= -cx, & g \cdot gx &= -cgx, \\ x \cdot 1 &= -c(1 + c')gx, & x \cdot g &= c(1 - c')x, & x \cdot x &= 0, & x \cdot gx &= 0, \\ gx \cdot 1 &= c(1 - c')gx, & gx \cdot g &= -c(1 + c')x, & gx \cdot x &= 0, & x \cdot gx &= 0, \end{aligned}$$

for any $0 \neq c', c \in k$.

Then one can check that $H_{4A} \in_{H_4} \mathcal{MHYD}^{H_4}(A, id)$, i.e., a left-right (A, id) -Yetter-Drinfeld Hom-module over (H_4, α) .

Let $B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c'' & 0 \\ 0 & 0 & 0 & c'' \end{pmatrix}$. Similarly, define $(H_{4B}, \alpha) = (H_4, \alpha)$ as k -vector spaces with the right H_4 -Hom-comodule structure via Δ and the left H_4 -module structures as follows:

$$\begin{aligned} 1 \cdot 1 &= 1, & 1 \cdot g &= g, & 1 \cdot x &= cx, & 1 \cdot gx &= cgx \\ g \cdot 1 &= 1, & g \cdot g &= g, & g \cdot x &= -cx, & g \cdot gx &= -cgx \\ x \cdot 1 &= -c(1 + c'')gx, & x \cdot g &= c(-1 + c'')x, & x \cdot x &= 0, & x \cdot gx &= 0, \\ gx \cdot 1 &= c(-1 + c'')gx, & gx \cdot g &= -c(1 + c'')x, & gx \cdot x &= 0, & x \cdot gx &= 0, \end{aligned}$$

for any $0 \neq c'', c \in k$.

It is straightforward to see that $H_{4B} \in_{H_4} \mathcal{MHYD}^{H_4}(id, B)$, a left-right (id, B) -Yetter-Drinfeld Hom-module over (H_4, α) .

$$\text{Let } A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c' & 0 \\ 0 & 0 & 0 & c' \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c'' & 0 \\ 0 & 0 & 0 & c'' \end{pmatrix}.$$

Define $(H_{4A,B}, \alpha) = (H_4, \alpha)$ as k -vector spaces, with the right H_4 -Hom-comodule structure via Δ and the left H -module structures as follows:

$$\begin{aligned} 1 \cdot 1 &= 1, & 1 \cdot g &= g, & 1 \cdot x &= cx, & 1 \cdot gx &= cgx \\ g \cdot 1 &= 1, & g \cdot g &= g, & g \cdot x &= -cx, & g \cdot gx &= -cgx \\ x \cdot 1 &= -c(c' + c'')gx, & x \cdot g &= c(-c' + c'')x, & x \cdot x &= 0, & x \cdot gx &= 0, \\ gx \cdot 1 &= c(-c' + c'')gx, & gx \cdot g &= -c(c' + c'')x, & gx \cdot x &= 0, & gx \cdot gx &= 0, \end{aligned}$$

for any $0 \neq c'', c \in k$.

Then it is straightforward to see that $H_{4(A,B)}$ is a left-right (A, B) -Yetter-Drinfeld Hom-module over (H_4, α) , i.e., $H_{4(A,B)} \in {}_{H_4} \mathcal{MHYD}^{H_4}(A, B)$.

Definition 2.6. A left-right *monoidal Hom-entwining structure* is a triple (H, C, ψ) , where (H, α) is a monoidal Hom-algebra and (C, γ) is a monoidal Hom-coalgebra with a linear map $\psi : H \otimes C \rightarrow H \otimes C$, $h \otimes c \mapsto_\psi h \otimes c^\psi$ satisfying the following conditions:

$$\psi(hg) \otimes c^\psi =_\phi h_\psi g \otimes \gamma(\gamma^{-1}(c)^{\psi\phi}), \quad (2.3)$$

$$\psi 1 \otimes c^\psi = 1_A \otimes c, \quad (2.4)$$

$$\psi h \otimes \Delta(c^\psi) = \alpha(\phi_\psi \alpha^{-1}(h)) \otimes (c_1^\phi \otimes c_2^\psi), \quad (2.5)$$

$$\varepsilon(c^\psi)_\psi h = \varepsilon(c)a, \quad (2.6)$$

Over a monoidal Hom-entwining structure (H, C, ψ) , a left-right monoidal entwined Hom-module M is both a right C -Hom-comodule and a left H -Hom-module such that

$$\rho^M(h \cdot m) =_\psi \alpha^{-1}(h) \cdot m_{(0)} \otimes \alpha(m_{(1)})^\psi$$

for all $h \in H$ and $m \in M$. We denote the category of all monoidal entwined Hom-modules over (H, C, ψ) by ${}_H \mathcal{M}^C(\psi)$.

Let (H, α) be a monoidal Hom-Hopf algebra with S , and define a linear map

$$\psi(A, B) : H \otimes H \rightarrow H \otimes H, \quad a \otimes c \mapsto_\psi a \otimes c^\psi = \alpha^2(a_{21}) \otimes (B(a_{22})\alpha^{-2}(c))AS^{-1}(a_1),$$

for all $A, B \in \text{Aut}_{mHH}(H)$.

Proposition 2.7. With notations as above, $(H, H, \psi(A, B))$ is a monoidal Hom-entwining structure for all $A, B \in \text{Aut}_{mHH}(H)$.

Proof. We need to prove that Eqs.(2.3-2.6) hold. First, it is straightforward to check Eqs.(2.4) and (2.6). In what follows, we only verify Eqs.(2.3) and (2.5). In fact, for all

$a, b, c \in H$, we have

$$\begin{aligned}
& \phi a_\psi b \otimes \alpha(\alpha^{-1}(c)^{\psi\phi}) \\
&= \alpha^2(a_{21})_\psi b \otimes \alpha((B(a_{22})\alpha^{-2}(\alpha^{-1}(c)^\psi))AS^{-1}(a_1)) \\
&= \alpha^2(a_{21})\alpha^2(b_{21}) \otimes \alpha([B(a_{22})\alpha^{-2}((B(b_{22})\alpha^{-3}(c))AS^{-1}(b_1))]AS^{-1}(a_1)) \\
&= \alpha^2(a_{21}b_{21}) \otimes [B\alpha(a_{22})((B\alpha^{-1}(b_{22})\alpha^{-4}(c))AS^{-1}\alpha^{-1}(b_1))]AS^{-1}\alpha(a_1) \\
&= \alpha^2(a_{21}b_{21}) \otimes (B(a_{22}b_{22})\alpha^{-2}(c))(AS^{-1}(b_1)AS^{-1}(a_1)) =_\psi (ab) \otimes c^\psi,
\end{aligned}$$

and Eq.(2.3) is proven.

For all $a \in H$, we have

$$a_1 \otimes a_{211} \otimes a_{2121} \otimes a_{2122} \otimes a_{22} = \alpha(a_{11}) \otimes \alpha^{-1}(a_{12}) \otimes \alpha^{-2}(a_{21}) \otimes \alpha^{-1}(a_{221}) \otimes \alpha(a_{222}) \quad (2.7)$$

As for Eq.(2.5), we compute:

$$\begin{aligned}
& \alpha(\phi_\psi \alpha^{-1}(a)) \otimes (c_1^\phi \otimes c_2^\psi) \\
&= \alpha(\alpha^2((\psi \alpha^{-1}(a))_{21})) \otimes ((B((\psi \alpha^{-1}(a))_{22})\alpha^{-2}(c_1))AS^{-1}((\psi \alpha^{-1}(a))_1) \otimes c_2^\psi) \\
&= \alpha(\alpha^2(\alpha^2(\alpha^{-1}(a)_{21})_{21})) \otimes ((B((\alpha^2(\alpha^{-1}(a)_{21})_{22})\alpha^{-2}(c_1))AS^{-1}((\alpha^2(\alpha^{-1}(a)_{21})_1) \\
&\quad \otimes (B(\alpha^{-1}(a)_{22})\alpha^{-2}(c_2))AS^{-1}(\alpha^{-1}(a)_1)) \\
&= \alpha^4(a_{2121}) \otimes ((B\alpha(a_{2122})\alpha^{-2}(c_1))AS^{-1}\alpha(a_{211}) \otimes (B\alpha^{-1}(a_{22})\alpha^{-2}(c_2))AS^{-1}\alpha^{-1}(a_1)) \\
&\stackrel{(2.7)}{=} \alpha^2(a_{21}) \otimes ((B(a_{221})\alpha^{-2}(c_1))AS^{-1}(a_{12}) \otimes (B(a_{222})\alpha^{-2}(c_2))AS^{-1}(a_{11})) \\
&= \alpha^2(a_{21}) \otimes ((B(a_{22})\alpha^{-2}(c))AS^{-1}(a_1))_1 \otimes (B(a_{22})\alpha^{-2}(c))AS^{-1}(a_1))_2 \\
&= {}_\psi a \otimes \Delta(c^\psi).
\end{aligned}$$

and Eq.(2.5) is proven.

This finishes the proof. ■

By Proposition 2.7, we have a monoidal entwined Hom-module category ${}_H\mathcal{M}^H(\psi(A, B))$ over $(H, H, \psi(A, B))$ with $A, B \in \text{Aut}_{mHH}(H)$. In this case, for all $M \in {}_H\mathcal{M}^H(\psi(A, B))$, we have

$$\rho(h \cdot m) = \alpha(h_{21}) \cdot m_{(0)} \otimes (B(h_{22})\alpha^{-1}(m_{(1)}))A(S^{-1}(h_1)),$$

for all $h \in H, m \in M$. Thus means that ${}_H\mathcal{M}^H(\psi(A, B)) = {}_H\mathcal{MHYD}^H(A, B)$ as categories.

Definition 2.8. Let (H, α) be a monoidal Hom-algebra. A monoidal Hom-algebra (N, ν) is called an (H, α) -Hom-bicomodule algebra, if (N, ν) is a left (H, α) -Hom-comodule and a right (H, α) -Hom-comodule with coactions ρ_r and ρ_l obeying the following axioms:

- (1) $\rho_l(n) = n_{[-1]} \otimes n_{[0]}$, and $\rho_r(n) = n_{<0>} \otimes n_{<1>}$,
- (2) (N, ν) is a left H -Hom-comodule algebra;
- (3) (N, ν) is a right H -Hom-comodule algebra;
- (4) ρ_l and ρ_r satisfy the following compatibility condition: for all $n \in N$,
$$\begin{aligned}
& n_{<0>[-1]} \otimes n_{<0>[0]} \otimes \alpha^{-1}(n_{<1>}) = \alpha^{-1}(n_{[-1]}) \otimes n_{[0]<0>} \otimes n_{[0]<1>} \\
& = n_{\{-1\}} \otimes n_{\{0\}} \otimes n_{\{1\}} \in (H \otimes N) \otimes H = H \otimes (N \otimes H).
\end{aligned}$$

Example 2.9. Let $A, B \in \text{Aut}_{mHH}(H)$, and $H_{(A,B)} = H$ as algebra, with H -Hom-comodule structures as follows: for all $h \in H$,

$$\begin{aligned} H_{(A,B)} &\rightarrow H \otimes H_{(A,B)}, \quad h \mapsto h_{[-1]} \otimes h_{[0]} = A(h_1) \otimes h_2, \\ H_{(A,B)} &\rightarrow H_{(A,B)} \otimes H, \quad h \mapsto h_{<0>} \otimes h_{<1>} = h_1 \otimes B(h_2). \end{aligned}$$

Then one can check $H_{(A,B)}$ is an H -Hom-bimodule algebra.

Definition 2.10. Let (H, α) be a monoidal Hom-Hopf algebra and (N, ν) be an H -Hom-bicomodule algebra. A left-right Yetter-Drinfeld Hom-module is a k -modules (M, μ) together with a left N -action (denoted by $n \otimes m \mapsto n \cdot m$) and a right H -coaction (denoted by $m \mapsto m_0 \otimes m_1$) satisfying the equivalent compatibility conditions:

$$\begin{aligned} (n \cdot m)_0 \otimes (n \cdot m)_1 &= \nu(n_{[0]<0>}) \cdot m_0 \otimes (n_{[0]<1>} \alpha^{-1}(m_1)) S^{-1}(n_{[-1]}), \\ n_{<0>} \cdot m_0 \otimes n_{<1>} m_1 &= \mu((n_{[0]} \cdot \mu^{-1}(m))_0) \otimes (n_{[0]} \cdot \mu^{-1}(m))_1 n_{[-1]}. \end{aligned}$$

for all $n \in N$ and $m \in M$. Then we call (H, N, H) a Yetter-Drinfeld Hom-datum (H, N, H) (the second H is regarded as an H -Hom-bimodule coalgebra). Our notion for the category of a left-right Yetter-Drinfeld Hom-modules and N -linear H -colinear maps will be ${}_N\mathcal{MHYD}^H(H)$.

Example 2.11. Let $H_{(A,B)}$ be an H -Hom-cobimodule algebra, with an H -Hom-comodule structures shown in Example 2.9. Then we can consider the Yetter-Drinfeld Hom-datum $(H, H_{(A,B)}, H)$ and the Yetter-Drinfeld Hom-modules over it, ${}_{H_{(A,B)}}\mathcal{MHYD}^H(H)$.

Proposition 2.12. ${}_H\mathcal{MHYD}^H(A, B) = {}_{H_{(A,B)}}\mathcal{MHYD}^H(H)$.

It is easy to see that the compatibility conditions for the two categories are the same. The easy proof of this is left to the reader.

3. A BRAIDED T -CATEGORY $\mathcal{MHYD}(H)$

In this section, we will construct a class of new braided T -categories $\mathcal{MHYD}(H)$ over any monoidal Hom-Hopf algebra (H, α) .

Proposition 3.1. If $(M, \mu) \in {}_H\mathcal{MHYD}^H(A, B)$ and $(N, \nu) \in {}_H\mathcal{MHYD}^H(C, D)$, with $A, B, C, D \in \text{Aut}_{mHH}(H)$, then $(M \otimes N, \mu \otimes \nu) \in {}_H\mathcal{MHYD}^H(AC, DC^{-1}BC)$ with structures as follows:

$$\begin{aligned} h \cdot (m \otimes n) &= C(h_1) \cdot m \otimes C^{-1}BC(h_2) \cdot n, \\ m \otimes n &\mapsto (m_{(0)} \otimes n_{(0)}) \otimes n_{(1)} m_{(1)}. \end{aligned}$$

for all $m \in M, n \in N$ and $h \in H$.

Proof. First, it is easy to get that $(M \otimes N, \mu \otimes \nu)$ is a left H -module and a right H -comodule. Next, we compute the compatibility condition as follows:

$$\begin{aligned}
& (h \cdot (m \otimes n))_{(0)} \otimes (h \cdot (m \otimes n))_{(1)} \\
&= ((C(h_1) \cdot m)_{(0)} \otimes (C^{-1}BC(h_2) \cdot n)_{(0)}) \otimes (C^{-1}BC(h_2) \cdot n)_{(1)} C(h_1 \cdot m)_{(1)} \\
&\stackrel{(2.1)}{=} (C\alpha(h_{121}) \cdot m_{(0)} \otimes C^{-1}BC\alpha(h_{221}) \cdot n_{(0)}) \otimes [(DC^{-1}BC(h_{222})\alpha^{-1}(n_{(1)})) \\
&\quad CS^{-1}C^{-1}BC(h_{21})][(BC(h_{122})\alpha^{-1}(m_{(1)}))S^{-1}AC(h_{11})] \\
&= (C(h_{12}) \cdot m_{(0)} \otimes C^{-1}BC\alpha(h_{221}) \cdot n_{(0)}) \otimes [(DC^{-1}BC(h_{222})\alpha^{-1}(n_{(1)})) \\
&\quad S^{-1}BC\alpha(h_{212})][(BC(h_{211})\alpha^{-1}(m_{(1)}))S^{-1}AC(h_{11})] \\
&= (C(h_{12}) \cdot m_{(0)} \otimes C^{-1}BC\alpha(h_{221}) \cdot n_{(0)}) \otimes (DC^{-1}BC\alpha(h_{222})n_{(1)}) \\
&\quad [(BC\alpha^{-1}(S^{-1}(h_{212})h_{211})\alpha^{-1}(m_{(1)}))S^{-1}AC(h_{11})] \\
&= (C(h_{12}) \cdot m_{(0)} \otimes C^{-1}BC\alpha(h_{221}) \cdot n_{(0)}) \otimes (DC^{-1}BC\alpha(h_{222})n_{(1)}) \\
&\quad [(BC\alpha^{-1}(\varepsilon(h_{21})1_H)\alpha^{-1}(m_{(1)}))S^{-1}AC(h_{11})] \\
&= (C(h_{12})\varepsilon(h_{21}) \cdot m_{(0)} \otimes C^{-1}BC\alpha(h_{221}) \cdot n_{(0)}) \otimes (DC^{-1}BC\alpha(h_{222})n_{(1)}) \\
&\quad (m_{(1)}S^{-1}AC(h_{11})) \\
&= (C\alpha(h_{121})\varepsilon\alpha(h_{122}) \cdot m_{(0)} \otimes C^{-1}BC(h_{21}) \cdot n_{(0)}) \otimes (DC^{-1}BC(h_{22})n_{(1)}) \\
&\quad (m_{(1)}S^{-1}AC(h_{11})) \\
&= (C(h_{12}) \cdot m_{(0)} \otimes C^{-1}BC(h_{21}) \cdot n_{(0)}) \otimes (DC^{-1}BC(h_{22})(\alpha^{-1}(n_{(1)})\alpha^{-1}(m_{(1)}))) \\
&\quad S^{-1}AC\alpha(h_{11}) \\
&= (C\alpha(h_{211}) \cdot m_{(0)} \otimes C^{-1}BC\alpha(h_{212}) \cdot n_{(0)}) \otimes (DC^{-1}BC(h_{22})(\alpha^{-1}(n_{(1)}m_{(1)}))) \\
&\quad S^{-1}AC(h_1) \\
&= \alpha(h_{21}) \cdot (m \otimes n)_{(0)} \otimes DC^{-1}BC(h_{22})\alpha^{-1}(m \otimes n)_{(1)} AC(S^{-1}(h_1)).
\end{aligned}$$

for all $m \in M, n \in N$ and $h \in H$. This completes the proof. \blacksquare

Remark. 3.2. Note that, if $(M, \mu) \in {}_H\mathcal{MHYD}^H(A, B)$, $(N, \nu) \in {}_H\mathcal{MHYD}^H(C, D)$ and $(P, \varsigma) \in {}_H\mathcal{MHYD}^H(E, F)$, then $(M \otimes N) \otimes P = M \otimes (N \otimes P)$ as objects in ${}_H\mathcal{MHYD}^H(ACE, FE^{-1}DC^{-1}BCE)$.

Denote $G = \text{Aut}_{mHH}(H) \times \text{Aut}_{mHH}(H)$ a group with multiplication as follows: for all $A, B, C, D \in \text{Aut}_{mHH}(H)$,

$$(A, B) * (C, D) = (AC, DC^{-1}BC). \quad (3.1)$$

The unit of this group is (id, id) and $(A, B)^{-1} = (A^{-1}, AB^{-1}A^{-1})$.

The above proposition means that if $M \in {}_H\mathcal{MHYD}^H(A, B)$ and $N \in {}_H\mathcal{MHYD}^H(C, D)$, then $M \otimes N \in {}_H\mathcal{MHYD}^H((A, B) * (C, D))$.

Proposition 3.3. Let $(N, \nu) \in {}_H\mathcal{MHYD}^H(C, D)$ and $(A, B) \in G$. Define ${}^{(A, B)}N = N$ as vector space, with structures: for all $n \in N$ and $h \in H$.

$$h \triangleright n = C^{-1}BCA^{-1}(h) \cdot n,$$

$$n \mapsto n_{<0>} \otimes n_{<1>} = n_{(0)} \otimes AB^{-1}(n_{(1)}). \quad (3. 2)$$

Then

$${}^{(A,B)}N \in {}_H\mathcal{MHYD}^H(ACA^{-1}, AB^{-1}DC^{-1}BCA^{-1}) = {}_H\mathcal{MHYD}^H((A, B)*(C, D)*(A, B)^{-1}).$$

Proof. Obviously, the equations above define a module and a comodule action. In what follows, we show the compatibility condition:

$$\begin{aligned} & (h \triangleright n)_{<0>} \otimes (h \triangleright n)_{<1>} \\ &= (C^{-1}BCA^{-1}(h) \cdot n)_{(0)} \otimes AB^{-1}((C^{-1}BCA^{-1}(h) \cdot n)_{(1)}) \\ &= C^{-1}BCA^{-1}\alpha(h_{21}) \cdot n_{(0)} \otimes AB^{-1}((DC^{-1}BCA^{-1}(h_{22})\alpha^{-1}(n_{(1)})) \\ & \quad CC^{-1}BCA^{-1}S^{-1}(h_1)) \\ &= C^{-1}BCA^{-1}\alpha(h_{21}) \cdot n_{(0)} \otimes (AB^{-1}DC^{-1}BCA^{-1}(h_{22})AB^{-1}\alpha^{-1}(n_{(1)}))ACA^{-1}S^{-1}(h_1) \\ &= \alpha(h_{21}) \triangleright n_{<0>} \otimes (AB^{-1}DC^{-1}BCA^{-1}(h_{22})\alpha^{-1}(n_{<1>}))ACA^{-1}S^{-1}(h_1) \end{aligned}$$

for all $n \in N$ and $h \in H$, that is ${}^{(A,B)}N \in {}_H\mathcal{MHYD}^H(ACA^{-1}, AB^{-1}DC^{-1}BCA^{-1})$ ■

Remark. 3.4. Let $(M, \mu) \in {}_H\mathcal{MHYD}^H(A, B)$, $(N, \nu) \in {}_H\mathcal{MHYD}^H(C, D)$, and $(E, F) \in G$. Then by the above proposition, we have:

$${}^{(A,B)*}{}^{(E,F)}N = {}^{(A,B)}({}^{(E,F)}N),$$

as objects in ${}_H\mathcal{MHYD}^H(AECE^{-1}A^{-1}, AB^{-1}EF^{-1}DC^{-1}FE^{-1}BECE^{-1}A^{-1})$ and

$${}^{(E,F)}(M \otimes N) = {}^{(E,F)}M \otimes {}^{(E,F)}N,$$

as objects in ${}_H\mathcal{MHYD}^H(EACE^{-1}, EF^{-1}DC^{-1}BA^{-1}FACE^{-1})$.

Proposition 3.5. Let $(M, \mu) \in {}_H\mathcal{MHYD}^H(A, B)$ and $(N, \nu) \in {}_H\mathcal{MHYD}^H(C, D)$, take ${}^MN = {}^{(A,B)}N$ as explained in Subsection 1.2. Define a map $c_{M,N} : M \otimes N \rightarrow {}^MN \otimes M$ by

$$c_{M,N}(m \otimes n) = \nu(n_{(0)}) \otimes B^{-1}(n_{(1)}) \cdot \mu^{-1}(m). \quad (3. 3)$$

for all $m \in M, n \in N$. Then $c_{M,N}$ is both an H -module map and an H -comodule map, and satisfies the following formulae (for $(P, \varsigma) \in {}_H\mathcal{MHYD}^H(E, F)$):

$$a_{M \otimes N, P, M, N}^{-1} \circ c_{M \otimes N, P} \circ a_{M, N, P}^{-1} = (c_{M, N, P} \otimes id_N) \circ a_{M, N, P, N}^{-1} \circ (id_M \otimes c_{N, P}), \quad (3. 4)$$

$$a_{M, N, M, P} \circ c_{M, N \otimes P} \circ a_{M, N, P} = (id_{M, N} \otimes c_{M, P}) \circ a_{M, N, M, P} \circ (c_{M, N} \otimes id_P). \quad (3. 5)$$

Furthermore, if $(M, \mu) \in {}_H\mathcal{MHYD}^H(A, B)$ and $(N, \nu) \in {}_H\mathcal{MHYD}^H(C, D)$, then $c_{(E,F)M, (E,F)N} = c_{M, N}$, for all $(E, F) \in G$.

Proof. First, we prove that $c_{M,N}$ is an H -module map. Take $h \cdot (m \otimes n) = C(h_1) \cdot m \otimes C^{-1}BC(h_2) \cdot n$ and $h \cdot (n \otimes m) = C^{-1}BC(h_1) \cdot n \otimes B^{-1}DC^{-1}BC(h_2) \cdot m$ as explained

in Proposition 3.1.

$$\begin{aligned}
& c_{M,N}(h \cdot (m \otimes n)) \\
&= \nu((C^{-1}BC(h_2) \cdot n)_{(0)}) \otimes B^{-1}((C^{-1}BC(h_2) \cdot n)_{(1)}) \cdot \mu^{-1}(C(h_1) \cdot m) \\
&= \nu(C^{-1}BC\alpha(h_{221}) \cdot n_{(0)}) \otimes B^{-1}((DC^{-1}BC(h_{222})\alpha^{-1}(n_{(1)}))CC^{-1}BCS^{-1}(h_{21})) \\
&\quad \cdot \mu^{-1}(C(h_1) \cdot m) \\
&= \nu(C^{-1}BC\alpha(h_{221}) \cdot n_{(0)}) \otimes B^{-1}(DC^{-1}BC\alpha(h_{222})n_{(1)}) \\
&\quad \cdot ((CS^{-1}\alpha^{-1}(h_{21})C\alpha^{-2}(h_1)) \cdot \mu^{-1}(m)) \\
&= \nu(C^{-1}BC(h_{21}) \cdot n_{(0)}) \otimes B^{-1}(DC^{-1}BC(h_{22})n_{(1)}) \\
&\quad \cdot (C\alpha^{-1}((S^{-1}(h_{12})h_{11})) \cdot \mu^{-1}(m)) \\
&= \nu(C^{-1}BC(h_{21}) \cdot n_{(0)}) \otimes B^{-1}(DC^{-1}BC(h_{22})n_{(1)}) \cdot (C\alpha^{-1}(\varepsilon(h_1)1_H) \cdot \mu^{-1}(m)) \\
&= C^{-1}BC(h_1) \cdot \nu(n_{(0)}) \otimes B^{-1}DC^{-1}BC(h_2) \cdot (B^{-1}(n_{(1)}) \cdot \mu^{-1}(m)) \\
&= \psi_{N \otimes M}((h \otimes (\nu(n_{(0)}) \otimes B^{-1}(n_{(1)}) \cdot \mu^{-1}(m)))) \\
&= \psi_{N \otimes M} \circ (id \otimes c_{M,N})(h \otimes (m \otimes n))
\end{aligned}$$

Secondly, we check that $c_{M,N}$ is an H -comodule map as follows:

$$\begin{aligned}
& \rho_{N \otimes M} \circ c_{M,N}(m \otimes n) \\
&= ((\nu(n_{(0)}))_{<0>} \otimes (B^{-1}(n_{(1)}) \cdot \mu^{-1}(m))_{(0)}) \otimes (B^{-1}(n_{(1)}) \cdot \mu^{-1}(m))_{(1)} \\
&\quad (\nu(n_{(0)}))_{<1>} \\
&= (\nu(n_{(0)(0)}) \otimes (B^{-1}(n_{(1)}) \cdot \mu^{-1}(m))_{(0)}) \otimes (B^{-1}(n_{(1)}) \cdot \mu^{-1}(m))_{(1)} \\
&\quad AB^{-1}\alpha(n_{(0)(1)}) \\
&= (\nu(n_{(0)(0)}) \otimes B^{-1}\alpha(n_{(1)21}) \cdot \mu^{-1}(m)_{(0)}) \otimes ((BB^{-1}(n_{(1)22})\alpha^{-2}(m_{(1)})) \\
&\quad AB^{-1}S^{-1}(n_{(1)1}))AB^{-1}\alpha(n_{(0)(1)}) \\
&\stackrel{(1.7)}{=} (n_{(0)} \otimes B^{-1}\alpha(n_{(1)21}) \cdot \mu^{-1}(m_{(0)})) \otimes (\alpha(n_{(1)22})\alpha^{-1}(m_{(1)})) \\
&\quad (AB^{-1}S^{-1}\alpha(n_{(1)12})AB^{-1}\alpha(n_{(1)11})) \\
&= (n_{(0)} \otimes B^{-1}\alpha(n_{(1)21}) \cdot \mu^{-1}(m_{(0)})) \otimes (\alpha^2(n_{(1)22})m_{(1)})\varepsilon(n_{(1)1}) \\
&\stackrel{(1.7)}{=} (\nu(n_{(0)(0)}) \otimes B^{-1}(n_{(0)(1)}) \cdot \mu^{-1}(m_{(0)})) \otimes n_{(1)}m_{(1)} \\
&= (c_{M,N} \otimes id)((m_{(0)} \otimes n_{(0)}) \otimes n_{(1)}m_{(1)}) = (c_{M,N} \otimes id)\rho(m \otimes n).
\end{aligned}$$

Then we will check Eqs.(3.4) and (3.5). On the one hand,

$$\begin{aligned}
& a_{M \otimes N, P, M, N}^{-1} \circ c_{M \otimes N, P} \circ a_{M, N, P}^{-1}(m \otimes (n \otimes p)) \\
&= a_{M \otimes N, P, M, N}^{-1} \circ c_{M \otimes N, P}((\mu^{-1}(m) \otimes n) \otimes \varsigma(p)) \\
&= a_{M \otimes N, P, M, N}^{-1}(\varsigma^2(p_{(0)}) \otimes C^{-1}B^{-1}CD^{-1}\alpha(p_{(1)}) \cdot (\mu^{-2}(m) \otimes \nu^{-1}(n))) \\
&= (\varsigma(p_{(0)}) \otimes B^{-1}CD^{-1}\alpha(p_{(1)1}) \cdot \mu^{-2}(m)) \otimes D^{-1}\alpha^2(p_{(1)2}) \cdot n \\
&= (\varsigma^2(p_{(0)(0)}) \otimes B^{-1}CD^{-1}\alpha(p_{(0)(1)}) \cdot \mu^{-2}(m)) \otimes D^{-1}\alpha(p_{(1)}) \cdot n \\
&= (\varsigma(\varsigma(p_{(0)})_{<0>}) \otimes B^{-1}(\varsigma(p_{(0)})_{<1>}) \cdot \mu^{-2}(m)) \otimes D^{-1}\alpha(p_{(1)}) \cdot n \\
&= (c_{M, N, P} \otimes id_N)((\mu^{-1}(m) \otimes \varsigma(p_{(0)})) \otimes D^{-1}\alpha(p_{(1)}) \cdot n) \\
&= (c_{M, N, P} \otimes id_N) \circ a_{M, N, P, N}^{-1} \circ (id_M \otimes c_{N, P})(m \otimes (n \otimes p))
\end{aligned}$$

On the another hand,

$$\begin{aligned}
& a_{M, N, P, M} \circ c_{M, N \otimes P} \circ a_{M, N, P}((m \otimes n) \otimes p) \\
&= a_{M, N, P, M} \circ c_{M, N \otimes P}(\mu(m) \otimes (n \otimes \varsigma^{-1}(p))) \\
&= a_{M, N, P, M}((\nu \otimes \varsigma)(n \otimes \varsigma^{-1}(p))_{(0)} \otimes B^{-1}((n \otimes \varsigma^{-1}(p))_{(1)}) \cdot \mu^{-1}\mu(m)) \\
&= \nu^2(n_{(0)}) \otimes (p_{(0)} \otimes B^{-1}\alpha^{-1}(\alpha^{-1}(p_{(1)})n_{(1)}) \cdot \alpha^{-1}(m)) \\
&= \nu^2(n_{(0)}) \otimes \varsigma(\varsigma^{-1}(p)_{(0)}) \otimes (B^{-1}(\varsigma^{-1}(p)_{(1)}) \cdot (B^{-1}\alpha^{-1}(n_{(1)}) \cdot \alpha^{-1}(m))) \\
&= (id_{M, N} \otimes c_{M, P})((\nu^2(n_{(0)}) \otimes B^{-1}(n_{(1)}) \cdot \mu^{-1}(m)) \otimes p) \\
&= (id_{M, N} \otimes c_{M, P}) \circ a_{M, N, M, P} \circ (c_{M, N} \otimes id_P)((m \otimes n) \otimes p)
\end{aligned}$$

The proof is completed. ■

Lemma 3.6. The map $c_{M, N}$ defined by $c_{M, N}(m \otimes n) = \nu(n_{(0)}) \otimes B^{-1}(n_{(1)}) \cdot \mu^{-1}(m)$ is bijective; with inverse

$$c_{M, N}^{-1}(n \otimes m) = B^{-1}(S(n_{(1)})) \cdot \mu^{-1}(m) \otimes \nu(n_{(0)}).$$

Proof. First, we prove $c_{M, N}c_{M, N}^{-1} = id$. For all $m \in M, n \in N$, we have

$$\begin{aligned}
& c_{M, N}c_{M, N}^{-1}(n \otimes m) \\
&= c_{M, N}(B^{-1}S(n_{(1)}) \cdot \mu^{-1}(m) \otimes \nu(n_{(0)})) \\
&= \nu(\nu(n_{(0)})_{(0)}) \otimes B^{-1}(\nu(n_{(0)})_{(1)}) \cdot \mu^{-1}(B^{-1}S(n_{(1)}) \cdot \mu^{-1}(m)) \\
&= \nu^2(n_{(0)(0)}) \otimes B^{-1}((n_{(0)(1)})S\alpha^{-1}(n_{(1)})) \cdot \mu^{-1}(m) \\
&= \nu(n_{(0)}) \otimes B^{-1}((n_{(1)1})S(n_{(1)2})) \cdot \mu^{-1}(m) \\
&= \nu(n_{(0)}) \otimes B^{-1}(\varepsilon(n_{(1)})1_H) \cdot \mu^{-1}(m) \\
&= \nu(n_{(0)}) \otimes \varepsilon(n_{(1)})m = n \otimes m
\end{aligned}$$

The fact that $c_{M, N}^{-1}c_{M, N} = id$ is similar. This completes the proof. ■

Let H be a monoidal Hom-Hopf algebra and $G = Aut_{mHH}(H) \times Aut_{mHH}(H)$. Define $\mathcal{MHYD}(H)$ as the disjoint union of all ${}_H\mathcal{MHYD}^H(A, B)$ with $(A, B) \in G$. If we endow

$\mathcal{MHYD}(H)$ with tensor product shown in Proposition 3.1, then $\mathcal{MHYD}(H)$ becomes a monoidal category with unit k .

Define a group homomorphism $\varphi : G \rightarrow \text{Aut}(\mathcal{MHYD}(H))$, $(A, B) \mapsto \varphi(A, B)$ on components as follows:

$$\begin{aligned}\varphi_{(A,B)} : {}_H\mathcal{MHYD}^H(C, D) &\rightarrow {}_H\mathcal{MHYD}^H((A, B) * (C, D) * (A, B)^{-1}), \\ \varphi_{(A,B)}(N) &= {}^{(A,B)}N,\end{aligned}$$

and the functor $\varphi_{(A,B)}$ acts as identity on morphisms.

The braiding in $\mathcal{MHYD}(H)$ is given by the family $\{c_{M,N}\}$ in Proposition 3.5. So we get the following main theorem of this article.

Theorem 3.7. $\mathcal{MHYD}(H)$ is a braided T -category over G .

We will end this paper by giving some examples to illustrate the main theorem.

Example 3.8. For $\alpha = id_H$, we have $\mathcal{MHYD}(H) = \mathcal{YD}(H)$, the main constructions by Panaite and Staic [11].

Example 3.9. Let $(H_4 = k\{1, g, x, gx\}, \alpha, \Delta, \varepsilon, S)$ be the monoidal Hom-Hopf algebra (see Example 2.5 (2)). Let $\mathcal{MHYD}(H_4)$ be the disjoint union of all categories ${}_{H_4}\mathcal{MHYD}^{H_4}(A, B)$ of left-right Hom- (A, B) -Yetter-Drinfeld modules with $\text{Aut}_{mHH}(H_4) \times \text{Aut}_{mHH}(H_4)$. Then by Example 2.5, Proposition 3.3, Proposition 3.5 and Theorem 3.7, $\mathcal{MHYD}(H_4)$ is a new braided T -category over $\text{Aut}_{mHH}(H_4) \times \text{Aut}_{mHH}(H_4)$.

Explicitly, it is easily know that we have a group isomorphism: $\text{Aut}_{mHH}(H_4) = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix} \mid 0 \neq \lambda \in k \right\} \cong (k \setminus \{0\}, \times)$. Furthermore, one has: $\text{Aut}_{mHH}(H_4) \times \text{Aut}_{mHH}(H_4) \cong (k \setminus \{0\} \oplus k \setminus \{0\}, \times)$.

Let $H_{4A} \in {}_{H_4}\mathcal{MHYD}^{H_4}(A, id)$ and $H_{4B} \in {}_{H_4}\mathcal{MHYD}^{H_4}(id, B)$, for all $A, B \in \text{Aut}_{mHH}(H_4)$. Then $(H_{4A} \otimes H_{4B}, \alpha \otimes \alpha) \in {}_{H_4}\mathcal{MHYD}^{H_4}(A, B)$ with structures as follows:

$$h \cdot (x \otimes y) = h_1 \cdot x \otimes h_2 \cdot y,$$

for all $h \in H_4$, $x \in H_{4A}$, $y \in H_{4B}$.

If $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c' & 0 \\ 0 & 0 & 0 & c' \end{pmatrix}$ and $H_{4B} \in {}_{H_4}\mathcal{MHYD}^{H_4}(id, B)$, then ${}^{(A, id)}H_{4B} = H_{4B}$ as

vector spaces, with action \triangleright given by

$$\begin{aligned} 1 \triangleright 1 &= 1, & 1 \triangleright g &= g, & 1 \triangleright x &= cx, & 1 \triangleright gx &= cgx \\ g \triangleright 1 &= 1, & g \triangleright g &= g, & g \triangleright x &= -cx, & g \triangleright gx &= -cgx \\ x \triangleright 1 &= -c(1 + c'')/c'gx, & x \triangleright g &= c(-1 + c'')/c'x, & x \triangleright x &= 0, & x \triangleright gx &= 0, \\ gx \triangleright 1 &= c(-1 + c'')/c'gx, & gx \triangleright g &= -c(1 + c'')/c'x, & gx \triangleright x &= 0, & gx \triangleright gx &= 0, \end{aligned}$$

and coaction ρ_r defined by

$$\begin{aligned} \rho_r(1) &= 1 \otimes 1, & \rho_r(g) &= g \otimes g, \\ \rho_r(x) &= c^{-1}(x \otimes 1) + c^{-1}c'(g \otimes x), & \rho_r(gx) &= c^{-1}(gx \otimes g) + c^{-1}c'(1 \otimes gx), \end{aligned}$$

for all $c, c', c'' \in k \setminus \{0\}$, and ${}^{(A, id)}H_{4B} \in {}_{H_4} \mathcal{MHYD}^{H_4}(id, B)$.

Let $H_{4A} \in {}_{H_4} \mathcal{MHYD}^{H_4}(A, id)$ and $H_{4B} \in {}_{H_4} \mathcal{MHYD}^{H_4}(id, B)$. Then the braiding

$$c_{H_{4A}, H_{4B}} : H_{4A} \otimes H_{4B} \rightarrow {}^{(A, id)}H_{4B} \otimes H_{4A}$$

is given by

$$c_{H_{4A}, H_{4B}}(m \otimes n) = \alpha(n_1) \otimes n_2 \cdot \mu^{-1}(m),$$

for all $m \in H_{4A}, n \in H_{4B}$.

If we consider a system of the basis of $H_{4A} \otimes H_{4B}$: $\{1 \otimes 1, 1 \otimes g, 1 \otimes x, 1 \otimes gx, g \otimes 1, g \otimes g, g \otimes x, g \otimes gx, x \otimes 1, x \otimes g, x \otimes x, x \otimes gx, gx \otimes 1, gx \otimes g, gx \otimes x, gx \otimes gx\}$ which is denoted by $\{e_1, e_2, \dots, e_{16}\}$, then the braiding $c_{H_{4A}, H_{4B}}$ can be represented by the following matrix:

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & m & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & n & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & m & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

i.e. we have

$$c_{H_{4A}, H_{4B}}((e_1, e_2, \dots, e_{16})^T) = D(e_1, e_2, \dots, e_{16})^T,$$

for all $m \neq -1, n \neq 1, m, n \in k$.

Then by Theorem 3.7, $\mathcal{MHYD}(H_4)$ is a new Braided T -category over $(k/\{0\} \oplus k/\{0\}, \times)$.

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